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EMBEDDINGS OF REDUCED FREE PRODUCTS OF OPERATOR ALGEBRAS

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Abstract. Given reduced amalgamated free products of C*–algebras $(A,\phi)=\underset{\iota\in I}{*}(A_\iota,\phi_\iota)$ and $(D,\psi)=\underset{\iota\in I}{*}(D_\iota,\psi_\iota)$, an embedding $A\hookrightarrow D$ is shown to exist assuming there are conditional expectation preserving embeddings $A_\iota\hookrightarrow D_\iota$. This result is extended to show the existence of the reduced amalgamated free product of certain classes of unital completely positive maps. Finally, the reduced amalgamated free product of von Neumann algebras is defined in the general case and analogues of the above mentioned results are proved for von Neumann algebras.

Introduction.

We begin with some standard facts about freeness in groups, analogues of which we will consider in C*–algebras. If H is a subgroup of a group G and if G_ι is a subgroup of G containing H for every ι in some index set I, let us say that the family $(G_\iota)_{\iota\in I}$ is *free over* H (or *free with amalgamation over* H) if $g_1g_2\cdots g_n\notin H$ whenever $g_j\in G_{\iota_j}\backslash H$ for some $\iota_j\in I$ with $\iota_1\neq\iota_2$, $\iota_2\neq\iota_3,\ldots,\iota_{n-1}\neq\iota_n$. For example, if

$$G=\;(*_H)_{\iota\in I}G_\iota \tag{1}$$

is the amalgamated free product of groups, then the family $(G_\iota)_{\iota\in I}$ of subgroups is free over their common subgroup H. The amalgamated free product of groups (1) has the universal property that if K is any group and if $\pi_\iota:G_\iota\to K$ are group homomorphisms ($\iota\in I$) such that the restriction $\pi_\iota\restriction_H$ is the same for all $\iota\in I$ then there is a group homomorphism $\pi:G\to K$ such that $\pi\restriction_{G_\iota}=\pi_\iota$ for every $\iota\in I$. If, moreover, each of the homomorphisms π_ι is injective and if the family $\big(\pi_\iota(G_\iota)\big)_{\iota\in I}$ of images is free over $\pi(H)$ then the homomorphism π is injective on G.

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Given a unital C*–algebra B and for each ι in some index set I a unital C*–algebra A_ι containing B as a unital C*–subalgebra,[2] there is a universal amalgamated free product C*–algebra [**3**] $A = (*_B)_{\iota\in I} A_\iota$ of the A_ι over B, satisfying a suitable universal property like that for the amalgamated free product of groups. If G is the amalgamated free product of groups (1) taken with discrete topology, then the full group C*–algebra $C^*(G)$ is the universal amalgamated free product of full group C*–algebras $C^*(G_\iota)$ over their common C*–subalgebra $C^*(H)$.

There is a *reduced* amalgamated free product of C*–algebras, invented by Voiculescu [**27**], (and independently and less generally by Avitzour [**2**]), which is related to the amalgamated free product of groups via the reduced group C*–algebra instead of via the full group C*–algebra. This reduced amalgamated free product construction is natural in the context of Voiculescu's noncommutative probabilistic notion of freeness, which is in turn an abstraction in the context of operator algebras of the phenomenon of freeness in groups. If A is a unital C*–algebra having a unital C*–subalgebra B and a conditional expectation $\phi : A \to B$, and if A_ι is an intermediate C*–subalgebra, $B \subseteq A_\iota \subseteq A$, for every ι in some index set I, then the family $(A_\iota)_{\iota\in I}$ is said to be *free* with respect to ϕ if $\phi(a_1a_2\cdots a_n) = 0$ whenever $a_j \in A_{\iota_j} \cap \ker\phi$ for some $\iota_j \in I$ with $\iota_1 \neq \iota_2$, $\iota_2 \neq \iota_3, \ldots, \iota_{n-1} \neq \iota_n$. The motivating example is for the reduced C*–algebra $A = C^*_r(G)$ of a group G arising as an amalgamated free product of groups as in (1) and taken with discrete topology; thus A is the closed linear span of the image of the left regular representation $\lambda : G \to \mathcal{L}(\ell^2(G))$; letting $B = \overline{\mathrm{span}}\lambda(H) \subseteq A$, letting $A_\iota = \overline{\mathrm{span}}\lambda(G_\iota) \subseteq A$, so that $B \cong C^*_r(H)$ and $A_\iota \cong C^*_r(G_\iota)$, and taking the conditional expectation $\tau^G_H : A \to B$ given by, for $g \in G$,

$$\tau^G_H(\lambda(g)) = \begin{cases} \lambda(g) & \text{if } g \in H \\ 0 & \text{if } g \notin H, \end{cases}$$

the family $(A_\iota)_{\iota\in I}$ is free with respect to τ^G_H.

Let us now describe Voiculescu's *reduced amalgamated free product* construction. If B is a unital C*–algebra, if I is a set and if, for every $\iota \in I$, A_ι is a unital C*–algebra containing B as a unital C*–subalgebra and having a conditional expectation $\phi_\iota : A_\iota \to B$ whose GNS representation[3] is faithful, then there is a unital C*–algebra A containing B as a unital C*–subalgebra, having a conditional expectation $\phi : A \to B$ and having embeddings (i.e. injective

[2]By a *unital C*–subalgebra* of a C*–algebra A we will mean a C*–subalgebra B of A containing the identity element of A.

[3]The GNS representation of ϕ_ι is the canonical representation of A_ι as bounded adjointable operators on the Hilbert B–module $L^2(A_\iota, \phi_\iota)$

$*$–homomorphisms) $A_\iota \hookrightarrow A$ that restrict to the identity map on B, this setup being unique such that the following hold:

(i) $\phi\restriction_{A_\iota} = \phi_\iota$, for every $\iota \in I$,

(ii) the family $(A_\iota)_{\iota\in I}$ is free with respect to ϕ,

(iii) A is the C*–algebra generated by $\bigcup_{\iota\in I} A_\iota$,

(iv) the GNS representation of ϕ is faithful (on A).

We denote this reduced amalgamated free product by

$$(A,\phi) = \mathop{*}_{\iota\in I}(A_\iota,\phi_\iota).$$

If the C*–algebra over which one amalgamates is the scalars, $B = \mathbf{C}$, then the ϕ_ι are states and the construction is called simply the reduced free product. For the actual construction, see [**27**]. Details of Voiculescu's construction are reviewed in [**16**, §1], and we will in this note abide by the notation used there.

If we consider for the moment only the case when $B = \mathbf{C}$, i.e. simply the reduced free product of C*–algebras, we may ask: to what extent does the reduced free product of C*–algebras have a universal property, analogous to those for the free product of groups and the universal free product of C*–algebras? Since the reduced free product of C*–algebras frequently gives rise to simple C*–algebras, (see [**2**], [**19**], [**14**], [**15**] and [**10**]), it is clear that any universal property for the reduced free product should be quite a bit more restrictive in character than for the universal free product. At first glance, it seems plausible that the reduced free product could have the universal property, which could be called the universal property for state preserving and freeness preserving $*$–homomorphisms, that would be implied by a positive answer to the following question.

Question 1. *If*

$$(A,\phi) = \mathop{*}_{\iota\in I}(A_\iota,\phi_\iota)$$

is a reduced free product of C**–algebras, where the* ϕ_ι *are states on the unital C***–algebras* A_ι *having faithful GNS representations, and if* D *is a unital C***–algebra with a state* ψ *and with unital* $*$*–homomorphisms* $\pi_\iota : A_\iota \to D$ *such that*

(i) $\psi\circ\pi_\iota = \phi_\iota$ *for every* $\iota \in I$*,*

(ii) *the family* $\big(\pi_\iota(A_\iota)\big)_{\iota\in I}$ *is free with respect to* ψ*,*

does it follow that there is a $*$*–homomorphism* $\pi : A \to D$ *such that, denoting by* $\alpha_\iota : A_\iota \to A$ *the injective* $*$*–homomorphisms arising from the free product construction,* $\pi\circ\alpha_\iota = \pi_\iota$ *for every* $\iota \in I$*? (Note that* π *would necessarily be injective.)*

As observed in [**20**, 1.3], the answer to Question 1 is "yes" if the state ψ on D is assumed to be faithful, (and a similar result holds in the amalgamated case). However, in [**20**, 1.4] an elementary example was given for which the answer to Question 1 is "no." Unfortunately, in the printed version of [**20**], notational errors (for which the authors of [**20**] take full responsibility) were introduced into Example 1.4. We therefore repeat this example below.

Example 2 ([**20**]). *If $0<\gamma<1$ we denote by*

$$\underset{\gamma}{\overset{p}{\mathbf{C}}} \oplus \underset{1-\gamma}{\mathbf{C}} \tag{2}$$

the pair (B,τ) where B is the two–dimensional C^*–algebra with a minimal projection p and where τ is the state on B satisfying $\tau(p)=\gamma$. Let

$$(A_1,\phi_1) = (\underset{3/4}{\overset{p}{\mathbf{C}}} \oplus \underset{1/4}{\mathbf{C}}),$$

$$(A_2,\phi_2) = (\underset{2/3}{\overset{q}{\mathbf{C}}} \oplus \underset{1/3}{\mathbf{C}})$$

and

$$(A,\phi) = (A_1,\phi_1) * (A_2,\phi_2).$$

Then q is Murray–von Neumann equivalent in A to a proper subprojection of p; (see [**14**, 2.7] *or* [**1**] *for this result). Let $D=M_2(A)$ and let ψ be the state on D given by $\psi(\begin{smallmatrix} b_{11} & b_{12} \\ b_{21} & b_{22}\end{smallmatrix}) = \phi(b_{11})$. Although ψ is not faithful, clearly the GNS representation of ψ is faithful on D. Let $\pi_j : A_j \to D$ be the unital *–homomorphisms such that $\pi_1(p) = \left(\begin{smallmatrix} p & 0 \\ 0 & 0\end{smallmatrix}\right)$ and $\pi_2(q) = \left(\begin{smallmatrix} q & 0 \\ 0 & 1\end{smallmatrix}\right)$. Then $\psi\circ\pi_j = \phi_j$. Moreover, the pair $\big(\pi_1(A_1),\pi_2(A_2)\big)$ is clearly free with respect to ψ. There cannot, however, be a *–homomorphism $\pi : A \to D$ such that $\pi(p)=\pi_1(p)$ and $\pi(q)=\pi_2(q)$, because, as can be seen from* [**14**, 2.7]*, $\pi_2(q)$ is not equivalent in D to a subprojection of $\pi_1(p)$.*

The main result of this paper is an embedding result (Theorem 1.3) implying the following.

Property 3. *Let*

$$(A,\phi) = \mathop{*}_{\iota\in I}(A_\iota,\phi_\iota)$$

be a reduced free product of C^*–algebras, where the ϕ_ι are states on the unital C^*–algebras A_ι having faithful GNS representations. Denote by $\alpha_\iota : A_\iota \to A$ the embeddings arising from the reduced free product construction. Suppose for every $\iota\in I$ there is a unital C^*–algebra D_ι with a state ψ_ι having faithful GNS representation, and suppose there is a $*$–homomorphism $\pi_\iota : A_\iota \to D_\iota$ such that $\psi_\iota\circ\pi_\iota = \phi_\iota$. Let

$$(D,\psi) = \mathop{*}_{\iota\in I}(D_\iota,\psi_\iota)$$

be the reduced free product of C–algebras and denote by $\delta_\iota : D_\iota \to D$ the embeddings arising from the reduced free product construction. Then there is a $*$–homomorphism $\pi : A \to D$ such that $\pi \circ \alpha_\iota = \delta_\iota \circ \pi_\iota$ for every $\iota \in I$.*

This property was previously known under the additional assumption that every ψ_ι is faithful, which by [**13**] implies that ψ is faithful on D; as noted after Question 1, this in turn implies the existence of π. Theorem 1.3 actually proves more generally a version of Property 3 for reduced amalgamated free products of C*–algebras. Such an embedding result is frequently useful for understanding reduced free product C*–algebras; it has been used in [**17**] and several times in [**16**].

We should point out that M. Choda has in [**9**] stated a theorem about reduced free products of completely positive maps which is more general than Property 3. However, her proof is incomplete, as it implicitly uses the full generality of Property 3 without justifying its validity. Therefore, a proof of Property 3 is called for.

In §1, the main theorem about embeddings of reduced amalgamated free products of C*–algebras is proved. In §2, Choda's argument proving the existence of reduced free products of state preserving completely positive maps is generalized to prove existence of reduced amalgamated free products of certain sorts of completely positive maps. In §3, we consider the reduced free product with amalgamation of von Neumann algebras and prove analogues of the results in §1 and §2 for von Neumann algebras.

§1. Embeddings.

In the following lemma, with the reduced amalgamated free product of C*–algebras $(A, \phi) = \underset{\iota\in I}{*} (A_\iota, \phi_\iota)$ we view each A_ι as a C*–subalgebra of A via the canonical embedding arising from the free product construction.

Lemma 1.1. *Let B be a unital C*–algebra, let I be a set and for every $\iota \in I$ let A_ι be a unital C*–algebra containing a copy of B as a unital C*–subalgebra and having a conditional expectation $\phi_\iota : A_\iota \to B$ whose GNS representation is faithful. Let*

$$(A, \phi) = \underset{\iota\in I}{*}(A_\iota, \phi_\iota)$$

be the reduced amalgamated free product. Then for every $\iota_0 \in I$ there is a conditional expectation $\Phi_{\iota_0} : A \to A_{\iota_0}$ such that $\Phi_{\iota_0}\restriction_{A_\iota} = \phi_\iota$ for every $\iota \in I\backslash\{\iota_0\}$ and $\Phi_{\iota_0}(a_1 a_2 \cdots a_n) = 0$ whenever $n \geq 2$ and $a_j \in A_{\iota_j} \cap \ker\phi$ with $\iota_1 \neq \iota_2, \ldots, \iota_{n-1} \neq \iota_n$.

Proof. We use the same notation as in [**16**] for the free product construction. Thus $E_\iota = L^2(A_\iota, \phi_\iota)$, $\xi_\iota = \widehat{1_{A_\iota}} \in E_\iota$, $E_\iota = \xi_\iota B \oplus E_\iota^{\mathrm{o}}$ and A acts on the Hilbert B–module

$$E = \xi B \oplus \bigoplus_{\substack{n\geq 1 \\ \iota_1,\dots,\iota_n\in I \\ \iota_1\neq\iota_2,\iota_2\neq\iota_3,\dots,\iota_{n-1}\neq\iota_n}} E_{\iota_1}^{\mathrm{o}} \otimes_B E_{\iota_2}^{\mathrm{o}} \otimes_B \cdots \otimes_B E_{\iota_n}^{\mathrm{o}}.$$

Identify the submodule $\xi B \oplus E_{\iota_0}^{\mathrm{o}}$ of E with the Hilbert B–module E_{ι_0} and let $Q_{\iota_0} : E \to E_{\iota_0}$ be the projection. Then $\Phi_{\iota_0}(x) = Q_{\iota_0} x Q_{\iota_0}$ has the desired properties.

□

Explication 1.2. Consider the GNS representation $\big(\sigma, L^2(A, \Phi_{\iota_0}), \eta\big) = \mathrm{GNS}(A, \Phi_{\iota_0})$ associated with the conditional expectation $\Phi_{\iota_0} : A \to A_{\iota_0}$ found in Lemma 1.1. Since A is the closed linear span of B and the set of reduced words of the form $a_1 a_2 \cdots a_n$ where $a_j \in A_{\iota_j} \cap \ker\phi$ and $\iota_j \neq \iota_{j+1}$, we see that the Hilbert A_{ι_0}–module in the GNS representation is

$$L^2(A, \Phi_{\iota_0}) = A_{\iota_0} \oplus \bigoplus_{\substack{n\geq 1 \\ \iota_1,\dots,\iota_n\in I \\ \iota_1\neq\iota_2,\dots,\iota_{n-1}\neq\iota_n \\ \iota_n\neq\iota_0}} E_{\iota_1}^{\mathrm{o}} \otimes_B \cdots \otimes_B E_{\iota_n}^{\mathrm{o}} \otimes_B A_{\iota_0}. \tag{3}$$

Moreover, the action σ of A on $L^2(A, \Phi_{\iota_0})$ is determined by its restrictions $\sigma\restriction_{A_\iota}$, which are easily described.

Let $\rho : A_{\iota_0} \to \mathcal{L}(\mathcal{V})$ be a unital $*$–homomorphism, for some Hilbert space $\mathcal{V}$. Then $\sigma \otimes 1 : A \to \mathcal{L}\big(L^2(A, \Phi_{\iota_0}) \otimes_\rho \mathcal{V}\big)$ is a $*$–homomorphism; it is the representation induced, in the sense of Rieffel [**25**], from ρ up to A, with respect to the conditional expectation Φ_{ι_0}, and we will denote this induced representation by $\rho\downharpoonleft^A$. We have the following explicit description of $\rho\downharpoonleft^A$, obtained by tensoring (3) with $\otimes_\rho \mathcal{V}$ on the right. Writing $\mathcal{H} = L^2(A, \Phi_{\iota_0}) \otimes_\rho \mathcal{V}$ we have

$$\mathcal{H} = \mathcal{V} \oplus \bigoplus_{\substack{n\geq 1 \\ \iota_1,\dots,\iota_n\in I \\ \iota_1\neq\iota_2,\dots,\iota_{n-1}\neq\iota_n \\ \iota_n\neq\iota_0}} E_{\iota_1}^{\mathrm{o}} \otimes_B \cdots \otimes_B E_{\iota_n}^{\mathrm{o}} \otimes_\rho \mathcal{V}. \tag{4}$$

Moreover, the $*$–homomorphism $\sigma \otimes 1$ is determined by its restrictions

$$\sigma_\iota \overset{\mathrm{def}}{=} (\sigma \otimes 1)\restriction_{A_\iota} : A_\iota \to \mathcal{L}(\mathcal{H}),$$

given as follows. Consider the Hilbert spaces

$$\mathcal{H}(\iota) = \begin{cases} (\eta_\iota B \otimes_{\rho\restriction_B} \mathcal{V}) \oplus \bigoplus\limits_{\substack{n\geq 1 \\ \iota_1,\dots,\iota_n\in I \\ \iota_1\neq\iota_2,\dots,\iota_{n-1}\neq\iota_n \\ \iota_n\neq\iota_0,\, \iota_1\neq\iota}} E^{\circ}_{\iota_1} \otimes_B \cdots \otimes_B E^{\circ}_{\iota_n} \otimes_\rho \mathcal{V} & \text{if } \iota \neq \iota_0 \\ \bigoplus\limits_{\substack{n\geq 1 \\ \iota_1,\dots,\iota_n\in I \\ \iota_1\neq\iota_2,\dots,\iota_{n-1}\neq\iota_n \\ \iota_n\neq\iota_0,\, \iota_1\neq\iota_0}} E^{\circ}_{\iota_1} \otimes_B \cdots \otimes_B E^{\circ}_{\iota_n} \otimes_\rho \mathcal{V} & \text{if } \iota = \iota_0, \end{cases}$$

where $\eta_\iota B$ is just the Hilbert B–module B with identity element denoted by η_ι. If $\iota \in I\backslash\{\iota_0\}$ let

$$W_\iota : E_\iota \otimes_B \mathcal{H}(\iota) \to \mathcal{H} \tag{5}$$

be the unitary defined, using the symbol $\ddot{\otimes}$ to denote the tensor product in (5), by

$$\begin{aligned} W_\iota : \xi_\iota \ddot{\otimes} (\eta_\iota \otimes v) &\mapsto v \\ \zeta \ddot{\otimes} (\eta_\iota \otimes v) &\mapsto \zeta \otimes v \\ \xi_\iota \ddot{\otimes} (\zeta_1 \otimes \cdots \otimes \zeta_n \otimes v) &\mapsto \zeta_1 \otimes \cdots \otimes \zeta_n \otimes v \\ \zeta \ddot{\otimes} (\zeta_1 \otimes \cdots \otimes \zeta_n \otimes v) &\mapsto \zeta \otimes \zeta_1 \otimes \cdots \otimes \zeta_n \otimes v \end{aligned}$$

whenever $v \in \mathcal{V}$, $\zeta \in E^{\circ}_\iota$, $\zeta_j \in E^{\circ}_{\iota_j}$ and $\iota \neq \iota_1$, $\iota_1 \neq \iota_2$, $\dots$, $\iota_{n-1} \neq \iota_n$, $\iota_n \neq \iota_0$. Then for every $\iota \in I\backslash\{\iota_0\}$ and $a \in A_\iota$, we have

$$\sigma_\iota(a) = W_\iota(a \otimes 1_{\mathcal{H}(\iota)})W_\iota^*.$$

Similarly, define the unitary

$$W_{\iota_0} : \mathcal{V} \oplus \big(E_\iota \otimes_B \mathcal{H}(\iota_0)\big) \to \mathcal{H}$$

by

$$\begin{aligned} W_{\iota_0} : v \oplus 0 &\mapsto v \\ 0 \oplus \big(\xi_{\iota_0} \ddot{\otimes} (\zeta_1 \otimes \cdots \otimes \zeta_n \otimes v)\big) &\mapsto \zeta_1 \otimes \cdots \otimes \zeta_n \otimes v \\ 0 \oplus \big(\zeta \ddot{\otimes} (\zeta_1 \otimes \cdots \otimes \zeta_n \otimes v)\big) &\mapsto \zeta \otimes \zeta_1 \otimes \cdots \otimes \zeta_n \otimes v. \end{aligned}$$

Then

$$\sigma_{\iota_0}(a) = W_{\iota_0}\big(\rho(a) \oplus (a \otimes 1_{\mathcal{H}(\iota_0)})\big)W_{\iota_0}^*.$$

Note that the above description is related to the construction of the conditionally free product, due to Bożejko and Speicher [**7**], (see also [**6**]).

Theorem 1.3. *Let $\widetilde{B}$ be a unital C^*–algebra. Let I be a set and for every $\iota \in I$ let $\widetilde{A}_\iota$ be a unital C^*–algebra containing a copy of $\widetilde{B}$ as a unital C^*–subalgebra and having a conditional expectation $\tilde{\phi}_\iota : \widetilde{A}_\iota \to \widetilde{B}$. Suppose that B is a unital C^*–algebra that is contained as a C^*–subalgebra of $\widetilde{B}$ (without necessarily containing the identity element of $\widetilde{B}$) and suppose that for every $\iota \in I$ A_ι is a unital C^*–algebra that is contained as a C^*–subalgebra of $\widetilde{A}_\iota$, that $B \subseteq A_\iota$, that B contains the identity element of A_ι and that $\tilde{\phi}_\iota(A_\iota) = B$. Let $\phi_\iota : A_\iota \to B$ be the restriction of $\tilde{\phi}_\iota$ and suppose that $\tilde{\phi}_\iota$ and ϕ_ι have faithful GNS representations. Let $\kappa_\iota : A_\iota \to \widetilde{A}_\iota$ denote the inclusion. Consider the reduced amalgamated free products of C^*–algebras*

$$(\widetilde{A}, \tilde{\phi}) = \mathop{*}_{\iota \in I} (\widetilde{A}_\iota, \tilde{\phi}_\iota)$$
$$(A, \phi) = \mathop{*}_{\iota \in I} (A_\iota, \phi_\iota)$$

and denote the inclusions arising from the free product construction by

$$\widetilde{\alpha}_\iota : \widetilde{A}_\iota \to \widetilde{A}$$
$$\alpha_\iota : A_\iota \to A.$$

Then there is a $*$–homomorphism $\kappa : A \to \widetilde{A}$ such that

$$\forall \iota \in I \qquad \kappa \circ \alpha_\iota = \widetilde{\alpha}_\iota \circ \kappa_\iota. \tag{6}$$

Moreover, κ is necessarily injective and is the unique $*$–homomorphism satisfying (6).

Proof. Since A is generated by $\bigcup_{\iota \in I} \alpha_\iota(A_\iota)$, it is clear that κ will be unique if it exists. Let 1 denote the identity element of $\widetilde{B}$ and let p be the identity element of B. If $p \neq 1$ then we may replace B by $B + \mathbf{C}(1-p)$ and each A_ι by $A_\iota + \mathbf{C}(1-p)$; hence we may without loss of generality assume that B is a unital C*–subalgebra of $\widetilde{B}$ and thus each A_ι is a unital C*–subalgebra of $\widetilde{A}_\iota$. Let

$$(\tilde{\pi}_\iota, \widetilde{E}_\iota, \tilde{\xi}_\iota) = \mathrm{GNS}(\widetilde{A}_\iota, \tilde{\phi}_\iota),$$
$$(\pi_\iota, E_\iota, \xi_\iota) = \mathrm{GNS}(A_\iota, \phi_\iota)$$

and

$$(\widetilde{E}, \tilde{\xi}) = \mathop{*}_{\iota \in I} (\widetilde{E}_\iota, \tilde{\xi}_\iota),$$
$$(E, \xi) = \mathop{*}_{\iota \in I} (E_\iota, \xi_\iota).$$

The inclusion κ_ι gives an inner product preserving isometry of Banach spaces $E_\iota \hookrightarrow \widetilde{E}_\iota$ sending ξ_ι to $\tilde{\xi}_\iota$, and we identify E_ι with this subspace of $\widetilde{E}_\iota$ and thereby E_ι^{o} with the subspace of $\widetilde{E}_\iota^{\mathrm{o}}$. This allows canonical identification of the tensor product module

$$E_{\iota_1}^{\mathrm{o}} \otimes_B \cdots \otimes_B E_{\iota_{p-1}}^{\mathrm{o}} \otimes_B \widetilde{E}_{\iota_p}^{\mathrm{o}} \otimes_{\widetilde{B}} \cdots \otimes_{\widetilde{B}} \widetilde{E}_{\iota_n}^{\mathrm{o}}$$

with a closed subspace of $\widetilde{E}_{\iota_1}^{\mathrm{o}} \otimes_{\widetilde{B}} \cdots \otimes_{\widetilde{B}} \widetilde{E}_{\iota_n}^{\mathrm{o}}$. Hence, we may and do identify E with the subspace

$$\tilde{\xi}B \oplus \bigoplus_{\substack{n\geq 1 \\ \iota_1,\ldots,\iota_n\in I \\ \iota_1\neq\iota_2,\ldots,\iota_{n-1}\neq\iota_n}} E_{\iota_1}^{\mathrm{o}} \otimes_B \cdots \otimes_B E_{\iota_n}^{\mathrm{o}}$$

of $\widetilde{E}$. Let $\mathfrak{A} = (*_B)_{\iota\in I} A_\iota$ be the universal algebraic free product with amalgamation over B. Let $\sigma : \mathfrak{A} \to \mathcal{L}(E)$, respectively $\widetilde{\sigma} : \mathfrak{A} \to \mathcal{L}(\widetilde{E})$, be the homomorphism extending the homomorphisms $\alpha_\iota : A_\iota \to \mathcal{L}(E)$, respectively $\widetilde{\alpha}_\iota \circ \kappa_\iota : A_\iota \to \mathcal{L}(\widetilde{E})$, $(\iota \in I)$. In particular, we have $\overline{\sigma(\mathfrak{A})} = A$. In order to show that κ exists, it will suffice to show that

$$\forall x \in \mathfrak{A} \qquad \|\widetilde{\sigma}(x)\| \leq \|\sigma(x)\|.$$

Note that the subspace E of $\widetilde{E}$ is invariant under $\widetilde{\sigma}(\mathfrak{A})$ and that the restriction of $\widetilde{\sigma}(\cdot)$ to E gives σ. This implies

$$\forall x \in \mathfrak{A} \qquad \|\widetilde{\sigma}(x)\| \geq \|\sigma(x)\|,$$

which will in turn imply that κ is injective, once it is known to exist. Let τ be a faithful representation of $\widetilde{B}$ on a Hilbert space $\mathcal{W}$. Consider the Hilbert space $\widetilde{E} \otimes_\tau \mathcal{W}$ and let $\widetilde{\lambda} : \mathcal{L}(\widetilde{E}) \to \mathcal{L}(\widetilde{E} \otimes_\tau \mathcal{W})$ be the $*$–homomorphism given by $\widetilde{\lambda}(x) = x \otimes 1_{\mathcal{W}}$. Then $\widetilde{\lambda}$ is faithful, and hence it will suffice to show that

$$\forall x \in \mathfrak{A} \qquad \|\widetilde{\lambda} \circ \widetilde{\sigma}(x)\| \leq \|\sigma(x)\|. \tag{7}$$

Our strategy will be to show that $\widetilde{\lambda} \circ \widetilde{\sigma}$ decomposes as a direct sum of subrepresentations, each of which is of the form $(\nu\restriction^A) \circ \sigma$, where $\nu\restriction^A$ is the $*$–representation of A induced from a representation ν of some A_ι.

Given $n \geq 1$ and $\iota_1, \ldots, \iota_n$ with $\iota_1 \neq \iota_2, \ldots, \iota_{n-1} \neq \iota_n$, and given $p \in \{1, 2, \ldots, n\}$, consider the Hilbert space

$$E_{\iota_1}^{\mathrm{o}} \otimes_B \cdots \otimes_B E_{\iota_{p-1}}^{\mathrm{o}} \otimes_B K_{\iota_p} \otimes_{\widetilde{B}} \widetilde{E}_{\iota_{p+1}}^{\mathrm{o}} \otimes_{\widetilde{B}} \cdots \otimes_{\widetilde{B}} \widetilde{E}_{\iota_n}^{\mathrm{o}} \otimes_\tau \mathcal{W} =$$

$$\stackrel{\text{def}}{=} \begin{pmatrix} E_{\iota_1}^{\mathrm{o}} \otimes_B \cdots \otimes_B E_{\iota_{p-1}}^{\mathrm{o}} \otimes_B \widetilde{E}_{\iota_p}^{\mathrm{o}} \otimes_{\widetilde{B}} \widetilde{E}_{\iota_{p+1}} \otimes_{\widetilde{B}} \cdots \otimes_{\widetilde{B}} \widetilde{E}_{\iota_n}^{\mathrm{o}} \otimes_\tau \mathcal{W} \\ \ominus\ E_{\iota_1}^{\mathrm{o}} \otimes_B \cdots \otimes_B E_{\iota_{p-1}}^{\mathrm{o}} \otimes_B E_{\iota_p}^{\mathrm{o}} \otimes_B \widetilde{E}_{\iota_{p+1}} \otimes_{\widetilde{B}} \cdots \otimes_{\widetilde{B}} \widetilde{E}_{\iota_n}^{\mathrm{o}} \otimes_\tau \mathcal{W} \end{pmatrix}.$$

Heuristically, K_ι takes the place of $\widetilde{E}_\iota \ominus E_\iota$, even when the latter does not make sense. Then

$$\widetilde{E}\otimes_\tau \mathcal{W} = (E\otimes_{\tau\restriction_B}\mathcal{W})\oplus \bigoplus_{\substack{n\geq 1 \\ \iota_1,\ldots,\iota_n\in I \\ \iota_1\neq\iota_2,\ldots,\iota_{n-1}\neq\iota_n \\ p\in\{1,2,\ldots,n\}}} E^{\mathrm{o}}_{\iota_1}\otimes_B\cdots\otimes_B E^{\mathrm{o}}_{\iota_{p-1}}\otimes_B K_{\iota_p}\otimes_{\widetilde{B}}\widetilde{E}^{\mathrm{o}}_{\iota_{p+1}}\otimes_{\widetilde{B}}\cdots\otimes_{\widetilde{B}}\widetilde{E}^{\mathrm{o}}_{\iota_n}\otimes_\tau\mathcal{W}.$$

As we mentioned earlier, $\widetilde{\sigma}(\mathfrak{A})E \subseteq E$ and $\widetilde{\sigma}(\cdot)\restriction_E = \sigma(\cdot)$, so $E\otimes_{\tau\restriction_B}\mathcal{W}$ is invariant under $\widetilde{\lambda}\circ\widetilde{\sigma}(\mathfrak{A})$, and

$$\forall x\in\mathfrak{A}\qquad \|\widetilde{\lambda}\circ\widetilde{\sigma}(x)\restriction_{E\otimes_\tau\mathcal{W}}\| = \|\sigma(x)\|.$$

Since $\widetilde{\pi}_\iota(A_\iota)E_\iota\subseteq E_\iota$, it is not difficult to check that for every $n\geq 1$ and for every $\iota_1,\ldots,\iota_n\in I$ with $\iota_1\neq\iota_2,\ldots,\iota_{n-1}\neq\iota_n$,

$$\begin{aligned}\widetilde{\mathcal{W}}(\iota_1,\ldots,\iota_n) &\overset{\text{def}}{=} \overline{\widetilde{\lambda}\circ\widetilde{\sigma}(\mathfrak{A})(K_{\iota_1}\otimes_{\widetilde{B}}\widetilde{E}^{\mathrm{o}}_{\iota_2}\otimes_{\widetilde{B}}\cdots\otimes_{\widetilde{B}}\widetilde{E}^{\mathrm{o}}_{\iota_n}\otimes_\tau\mathcal{W})} = \\ &= (K_{\iota_1}\otimes_{\widetilde{B}}\widetilde{E}^{\mathrm{o}}_{\iota_2}\otimes_{\widetilde{B}}\cdots\otimes_{\widetilde{B}}\widetilde{E}^{\mathrm{o}}_{\iota_n}\otimes_\tau\mathcal{W})\quad\oplus \\ &\quad\oplus\bigoplus_{\substack{q\geq 1 \\ \iota'_1,\ldots,\iota'_q\in I \\ \iota'_1\neq\iota'_2,\ldots,\iota'_{q-1}\neq\iota'_q \\ \iota'_q\neq\iota_1}} E^{\mathrm{o}}_{\iota'_1}\otimes_B\cdots\otimes_B E^{\mathrm{o}}_{\iota'_q}\otimes_B K_{\iota_1}\otimes_{\widetilde{B}}\widetilde{E}^{\mathrm{o}}_{\iota_2}\otimes_{\widetilde{B}}\cdots\otimes_{\widetilde{B}}\widetilde{E}^{\mathrm{o}}_{\iota_n}\otimes_\tau\mathcal{W}.\end{aligned}$$

Thus

$$\widetilde{E}\otimes_\tau\mathcal{W} = (E\otimes_{\tau\restriction_B}\mathcal{W})\oplus\bigoplus_{\substack{n\geq 1 \\ \iota_1,\ldots,\iota_n\in I \\ \iota_1\neq\iota_2,\ldots,\iota_{n-1}\neq\iota_n}}\widetilde{\mathcal{W}}(\iota_1,\ldots,\iota_n);$$

hence in order to prove the theorem it will suffice to show that for every choice of $\iota_1,\ldots\iota_n$,

$$\forall x\in\mathfrak{A}\qquad\|\widetilde{\lambda}\circ\widetilde{\sigma}(x)\restriction_{\widetilde{\mathcal{W}}(\iota_1,\ldots,\iota_n)}\|\leq\|\sigma(x)\|. \tag{8}$$

But letting $\mathcal{V} = K_{\iota_1}\otimes_{\widetilde{B}}\widetilde{E}^{\mathrm{o}}_{\iota_2}\otimes_{\widetilde{B}}\cdots\otimes_{\widetilde{B}}\widetilde{E}_{\iota_n}\otimes_\tau\mathcal{W}$, letting $\nu : A_{\iota_1}\to\mathcal{L}(\mathcal{V})$ be the $*$–homomorphism $\nu(a) = (\widetilde{\pi}_{\iota_1}(a)\otimes 1_{\widetilde{E}^{\mathrm{o}}_{\iota_2}\otimes_B\cdots\otimes_B\widetilde{E}^{\mathrm{o}}_{\iota_n}\otimes_\tau\mathcal{W}})\restriction_{\mathcal{V}}$, and appealing to Explication 1.2, it is straightforward to check that

$$\widetilde{\lambda}\circ\widetilde{\sigma}(\cdot)\restriction_{\widetilde{\mathcal{W}}(\iota_1,\ldots,\iota_n)} = (\nu\,\lfloor^A)\circ\sigma,$$

where $\nu\,\lfloor^A$ is the representation of A induced from ν with respect to the conditional expectation $\Phi_{\iota_1} : A\to A_{\iota_1}$ found in Lemma 1.1; this in turn implies (8).

$\square$

Remark 1.4. Let us consider for a moment Theorem 1.3 when the subalgebra B over which we amalgamate is the scalars, $\mathbf{C}$. When taking the reduced free product $(A,\phi) = \mathop{*}_{\iota\in I}(A_\iota,\phi_\iota)$ of

C*–algebras, the states ϕ_ι are required to have faithful GNS representation in order that the C*–algebras A_ι are embedded in the free product C*–algebra A. However, upon relaxing this condition to the case of completely general states ϕ_ι and performing the reduced free product construction, one obtains

$$\mathop{*}_{\iota\in I}(A_\iota,\phi_\iota) = \mathop{*}_{\iota\in I}((A_\iota/\ker\pi_\iota),\dot{\phi}_\iota),$$

where π_ι is the GNS representation of ϕ_ι and where $\dot{\phi}_\iota$ is the state induced on the quotient $A_\iota/\ker\pi_\iota$ by ϕ_ι. Thus the canonical $*$–homomorphism $\alpha_\iota : A_\iota \to A$ has the same kernel as π_ι.

As a caveat, we would like to point out that with this relaxed definition of reduced free product, the statement of Theorem 1.3 does not in general hold if one tolerates ϕ_ι with nonfaithful GNS representations. Indeed, if for some $\iota \in I$ $A_\iota = \mathbf{C}\oplus\mathbf{C}$ with ϕ_ι non–faithful, if $\widetilde{A}_\iota = M_2(\mathbf{C})$ with a unital embedding $\kappa_\iota : A_\iota \hookrightarrow \widetilde{A}_\iota$ and if $\tilde{\phi}_\iota$ is a state on $M_2(\mathbf{C})$ such that $\tilde{\phi}_\iota\circ\kappa_\iota = \phi_\iota$, then $\alpha_\iota : A_\iota \to A$ is not injective, while $\tilde{\alpha}_\iota\circ\kappa_\iota$ is injective. This shows that there can be no $*$–homomorphism $\kappa : A \to \widetilde{A}$ satisfying equation (6). However, there is no problem allowing the $\tilde{\phi}_\iota$ to have nonfaithful GNS representations, as long as the restrictions ϕ_ι are taken with faithful GNS representations.

§2. Completely positive maps.

M. Choda [**9**] gave an argument which, when combined with an embedding result like Property 3, proves that if $\theta_\iota : A_\iota \to D_\iota$ is a unital completely positive map between unital C*–algebras for every $\iota\in I$, if ϕ_ι and ψ_ι are states on A_ι and respectively D_ι, each having faithful GNS representation, and if $\psi_\iota\circ\theta_\iota = \phi_\iota$ then letting

$$\begin{aligned}(A,\phi) &= \mathop{*}_{\iota\in I}(A_\iota,\phi_\iota)\\ (D,\psi) &= \mathop{*}_{\iota\in I}(D_\iota,\psi_\iota)\end{aligned}$$

be the reduced free products of C*–algebras and denoting by $\alpha_\iota : A_\iota \to A$ and $\delta_\iota : D_\iota \to D$ the injective $*$–homomorphisms arising from the free product constructions, there is a unital completely positive map $\theta : A \to D$ such that $\theta\circ\alpha_\iota = \theta_\iota\circ\delta_\iota$ for every $\iota\in I$, and such that $\theta(a_1a_2\cdots a_n) = \theta(a_1)\theta(a_2)\cdots\theta(a_n)$ whenever $a_j \in \alpha_{\iota_j}(A_{\iota_j})\cap\ker\phi$ for some $\iota_j\in I$ with $\iota_1\neq\iota_2,\ldots,\iota_{n-1}\neq\iota_n$.

In this section, we generalize this argument of Choda's to the case of reduced amalgamated free products of C*–algebras. The generalization consists of, in essence, replacing Stinespring's dilation theorem for completely positive maps into bounded operators on a Hilbert space by

Kasparov's generalization [**21**] to the case of completely positive maps into the algebra of bounded adjointable operators on a Hilbert B–module (see alternatively the book [**22**]). We would like to point out that Theorem 2.2 is quite similar in appearance to analogous results of F. Boca [**4**], [**5**] about completely positive maps on universal amalgamated free products of C*–algebras. However, the universal and reduced free products of C*–algebras are quite different in character, and we do not believe that Boca's results can be used directly to prove Theorem 2.2.

Lemma 2.1. *Let A and B be C*–algebras, let E and $\widetilde{E}$ be Hilbert A–modules, let F and $\tilde{F}$ be Hilbert B–modules and let $v \in \mathcal{L}(E,\widetilde{E})$, $w \in \mathcal{L}(F,\tilde{F})$. Suppose $\pi : A \to \mathcal{L}(F)$ and $\widetilde{\pi} : A \to \mathcal{L}(\tilde{F})$ are $*$–homomorphisms and suppose that*

$$\forall a \in A \quad \forall \xi \in F \qquad w(\pi(a)\xi) = \widetilde{\pi}(a)w(\xi). \tag{9}$$

Let $E \otimes_\pi F$ and $\widetilde{E} \otimes_{\widetilde{\pi}} \tilde{F}$ be the interior tensor products. Then there is an element $v \otimes w \in \mathcal{L}(E \otimes_\pi F, \widetilde{E} \otimes_{\widetilde{\pi}} \tilde{F})$ such that

$$\forall \zeta \in E \quad \forall \xi \in F \qquad (v \otimes w)(\zeta \otimes \xi) = (v\zeta) \otimes (w\xi).$$

If, moreover, $\langle v(\zeta), v(\zeta)\rangle = \langle \zeta, \zeta\rangle$ for every $\zeta \in E$ and $\langle w(\xi), w(\xi)\rangle = \langle \xi, \xi\rangle$ for every $\xi \in F$ then $\langle v \otimes w(\eta), v \otimes w(\eta)\rangle = \langle \eta, \eta\rangle$ for every $\eta \in E \otimes_\pi F$.

Proof. That $v \otimes w$ is bounded is a standard argument (compare p. 42 of [**22**]). Then one sees $(v \otimes w)^* = v^* \otimes w^*$. The final statement follows using the polarization identity.

□

Theorem 2.2. *Let B be a unital C*–algebra, let I be a set and for every $\iota \in I$ let A_ι and D_ι be unital C*–algebras containing copies of B as unital C*–subalgebras and having conditional expectations $\phi_\iota : A_\iota \to B$, respectively $\psi_\iota : D_\iota \to B$, whose GNS representations are faithful. Suppose that for each $\iota \in I$ there is a unital completely positive map $\theta_\iota : A_\iota \to D_\iota$ that is also a B–B bimodule map and satisfies $\psi_\iota \circ \theta_\iota = \phi_\iota$. Let*

$$(A, \phi) = \mathop{*}_{\iota \in I} (A_\iota, \phi_\iota)$$
$$(D, \phi) = \mathop{*}_{\iota \in I} (D_\iota, \psi_\iota)$$

be the reduced amalgamated free products of C*–algebras and denote by $\alpha_\iota : A_\iota \to A$ and $\delta_\iota : D_\iota \to D$ the embeddings arising from the free product constructions. Then there is a unital completely positive map $\theta : A \to D$ satisfying

$$\forall \iota \in I \qquad \theta \circ \alpha_\iota = \delta_\iota \circ \theta_\iota \tag{10}$$

and

$$\theta(a_1 a_2 \cdots a_n) = \theta(a_1)\theta(a_2)\cdots\theta(a_n) \tag{11}$$

whenever $a_j \in \alpha_{\iota_j}(A_{\iota_j} \cap \ker\phi_{\iota_j})$ *and* $\iota_1 \neq \iota_2,\ \iota_2 \neq \iota_3,\ \ldots,\ \iota_{n-1} \neq \iota_n$.

Proof. Note first that the assumptions imply that each θ_ι is the identity map on B. Let

$$(\pi_\iota, E_\iota, \xi_\iota) = \mathrm{GNS}(D_\iota, \psi_\iota), \qquad (E, \xi) = \mathop{*}_{\iota\in I}(E_\iota, \xi_\iota).$$

(We will usually write simply $d\zeta$ instead of $\pi_\iota(d)\zeta$, when $d \in D_\iota$ and $\zeta \in E_\iota$.) Recall that then $E_\iota = \xi_\iota B \oplus E_\iota^{\mathrm{o}}$, that the action $\pi_\iota\restriction_B$ leaves E_ι^{o} globally invariant, and that

$$E = \xi B \oplus \bigoplus_{\substack{n\geq 1\\ \iota_1,\ldots,\iota_n\in I\\ \iota_1\neq\iota_2,\ldots,\iota_{n-1}\neq\iota_n}} E_{\iota_1}^{\mathrm{o}} \otimes_B \cdots \otimes_B E_{\iota_n}^{\mathrm{o}}.$$

Consider the Hilbert B–module $F_\iota = A_\iota \otimes_{\pi_\iota\circ\theta_\iota} E_\iota$ and the specified element $\eta_\iota = 1 \otimes \xi_\iota \in F_\iota$. Since θ_ι restricts to the identity map on B, in F_ι we have $b \otimes \zeta = 1 \otimes (b\zeta)$ for every $b \in B$ and $\zeta \in E$. Consider the unital $*$–homomorphism $\sigma_\iota : A_\iota \to \mathcal{L}(F_\iota)$ given by

$$\forall a', a \in A_\iota \quad \forall \zeta \in E_\iota \qquad \sigma_\iota(a')(a \otimes \zeta) = (a'a) \otimes \zeta,$$

(see for example page 48 of [**22**]). Consider the map $\rho_\iota : \mathcal{L}(F_\iota) \to B$ given by $\rho_\iota(x) = \langle \eta_\iota, x\eta_\iota\rangle$. If $x \in \mathcal{L}(F_\iota)$ and if $b_1, b_2 \in B$ then $\rho_\iota\big(\sigma_\iota(b_1)x\sigma_\iota(b_2)\big) = b_1\rho_\iota(x)b_2$. If we use σ_ι to identify B with $\sigma_\iota(B) \subseteq \mathcal{L}(F_\iota)$ then we have that $\rho_\iota : \mathcal{L}(F_\iota) \to B$ is a conditional expectation. Clearly $L^2(\mathcal{L}(F_\iota), \rho_\iota) \cong F_\iota$ and the GNS representation of ρ_ι is faithful on $\mathcal{L}(F_\iota)$. We have that $\rho_\iota \circ \sigma_\iota = \phi_\iota$ since

$$\rho_\iota \circ \sigma_\iota(a) = \langle 1 \otimes \xi_\iota, a \otimes \xi_\iota\rangle = \langle \xi_\iota, \theta_\iota(a)\xi_\iota\rangle = \psi_\iota \circ \theta_\iota(a) = \phi_\iota(a). \tag{12}$$

Let

$$(\mathcal{M}, \rho) = \mathop{*}_{\iota\in I}(\mathcal{L}(F_\iota), \rho_\iota)$$

be the reduced amalgamated free product of C*–algebras and let $\lambda_\iota : \mathcal{L}(F_\iota) \to \mathcal{M}$ be the embedding arising from the free product construction. Note that $\mathcal{M} \subseteq \mathcal{L}(F)$ where

$$(F, \eta) = \mathop{*}_{\iota\in I}(F_\iota, \eta_\iota).$$

By Theorem 1.3 there is a $*$–homomorphism $\sigma : A \to \mathcal{M}$ such that

$$\forall \iota \in I \qquad \sigma \circ \alpha_\iota = \lambda_\iota \circ \sigma_\iota.$$

Consider the operator $v_\iota : E_\iota \to F_\iota$ given by $\zeta \to 1 \otimes \zeta$, and note that $\langle v_\iota\zeta, v_\iota\zeta\rangle = \langle \zeta, \zeta\rangle$ for every $\zeta \in E_\iota$, hence $v_\iota(E_\iota^{\mathrm{o}}) \subseteq F_\iota^{\mathrm{o}}$. A calculation using e.g. Lemma 5.4 of [**22**] shows that there is a bounded operator $F_\iota \to E_\iota$ sending $a \otimes \zeta$ to $\theta_\iota(a)\zeta$, which is then the adjoint of v_ι. Hence $v_\iota \in \mathcal{L}(E_\iota, F_\iota)$, and clearly $v_\iota^* v_\iota = 1$. Since θ_ι is a left B–module map, we have for every $b \in B$ and $\zeta \in E_\iota$ that $v_\iota(b\zeta) = 1 \otimes (b\zeta) = b \otimes \zeta = b(v_\iota(\zeta))$. Therefore, taking direct sums of operators $v_{\iota_1} \otimes \cdots \otimes v_{\iota_n}$ given by Lemma 2.1, we get $v \in \mathcal{L}(E, F)$ such that $\langle v\zeta, v\zeta\rangle = \langle \zeta, \zeta\rangle$ for every $\zeta \in E$, $v\xi = \eta$ and

$$v(\zeta_1 \otimes \zeta_2 \otimes \cdots \otimes \zeta_n) = (v_{\iota_1}\zeta_1) \otimes (v_{\iota_2}\zeta_2) \otimes \cdots \otimes (v_{\iota_n}\zeta_n)$$

whenever $\zeta_j \in E_{\iota_j}^{\mathrm{o}}$, $\iota_1, \ldots, \iota_n \in I$ and $\iota_j \neq \iota_{j+1}$. Let $\theta : A \to \mathcal{L}(E)$ be the unital completely positive map $\theta(x) = v^*\sigma(x)v$.

We will show that (10) and (11) hold, which will furthermore imply that $\theta(A) \subseteq D$. In order to show (10), let $w_\iota : E \to E_\iota \otimes_B E(\iota)$ and $y_\iota : F \to F_\iota \otimes_B F(\iota)$ be the unitaries used in the free product constructions to define the embeddings α_ι and, respectively, λ_ι. Note that $v_\iota\big(E(\iota)\big) \subseteq F(\iota)$ and that $y_\iota v = (v_\iota \otimes v\restriction_{E(\iota)})w_\iota$. Furthermore, observe that for $a \in A_\iota$ and $\zeta \in E_\iota$,

$$\big(v_\iota^*\sigma_\iota(a)v_\iota\big)\zeta = v_\iota^*(a \otimes \zeta) = \theta_\iota(a)\zeta.$$

Hence for $a \in A_\iota$,

$$\begin{aligned}\theta \circ \alpha_\iota(a) = v^*\sigma \circ \alpha_\iota(a)v = v^*\lambda_\iota \circ \sigma_\iota(a)v = v^* y_\iota^*\big(\sigma_\iota(a) \otimes 1_{F(\iota)}\big)y_\iota v = \\ = w_\iota^*\big(v_\iota^*\sigma_\iota(a)v_\iota \otimes (v\restriction_{E(\iota)})^* v\restriction_{E(\iota)}\big)w_\iota = w_\iota^*(\theta_\iota(a) \otimes 1_{E(\iota)})w_\iota = \alpha_\iota \circ \theta_\iota(a).\end{aligned}$$

Now to show that (11) holds, consider $a_j \in A_{\iota_j} \cap \ker\phi_{\iota_j}$ for some $\iota_j \in I$ $(1 \leq j \leq n)$ with $\iota_j \neq \iota_{j+1}$. Henceforth we will omit to write the inclusions α_ι and δ_ι, thinking instead of each A_ι as a subalgebra of A and of each D_ι as a subalgebra of D. It is easy to see that

$$\begin{aligned}\theta_{\iota_1}(a_1)\cdots\theta_{\iota_n}(a_n)\xi = \widehat{\theta_{\iota_1}(a_1)} \otimes \cdots \otimes \widehat{\theta_{\iota_n}(a_n)} = \\ = v^*\big((a_1 \otimes \xi_{\iota_1}) \otimes \cdots \otimes (a_n \otimes \xi_{\iota_n})\big) = \theta(a_1\cdots a_n)\xi.\end{aligned} \tag{13}$$

Now consider an element $\zeta_1 \otimes \cdots \otimes \zeta_p \in E$, where $\zeta_j \in E_{k_j}^{\mathrm{o}}$ for some $k_j \in I$ with $k_j \neq k_{j+1}$.

Let $P_0 : E \to \xi B$ be the projection and for $\ell \in \mathbf{N}$ let

$$P_\ell : E \to \bigoplus_{\substack{\iota_1,\ldots,\iota_\ell \in I \\ \iota_1\neq\iota_2,\ldots,\iota_{\ell-1}\neq\iota_\ell}} E_{\iota_1}^{\mathrm{o}} \otimes \cdots \otimes E_{\iota_\ell}$$

be the projection. Taking adjoints and using (13), we see that

$$P_0\theta_{\iota_1}(a_1)\cdots\theta_{\iota_n}(a_n)(\zeta_1\otimes\cdots\otimes\zeta_p)=P_0\theta(a_1\cdots a_n)(\zeta_1\otimes\cdots\otimes\zeta_p).$$

Now letting $\ell\in\mathbf{N}$ we will use standard techniques (see, for example, [**18**] and [**16**]) to show that

$$P_\ell\theta_{\iota_1}(a_1)\cdots\theta_{\iota_n}(a_n)(\zeta_1\otimes\cdots\otimes\zeta_p)=P_\ell\theta(a_1\cdots a_n)(\zeta_1\otimes\cdots\otimes\zeta_p). \tag{14}$$

If $\ell>n+p$ or $\ell<|n-p|$ then it is clear that both sides of (14) are zero. If $\ell=n+p$ then both sides of (14) are zero unless $\iota_n\neq k_1$, in which case a calculation similar to (13) shows that (14) holds. Let $Q_\iota^{\mathrm{o}}:E_\iota\to E_\iota^{\mathrm{o}}$ and $R_\iota^{\mathrm{o}}:F_\iota\to F_\iota^{\mathrm{o}}$ be the projections, and note that $R_\iota^{\mathrm{o}}v_\iota=v_\iota Q_\iota^{\mathrm{o}}$. Consider when $n+p-\ell=1$. Then both sides of (14) are zero unless $\iota_n=k_1$, in which case

$$\begin{aligned}
&P_\ell\theta_{\iota_1}(a_1)\cdots\theta_{\iota_n}(a_n)(\zeta_1\otimes\cdots\otimes\zeta_p)=\\
&\quad=\widehat{\theta_{\iota_1}(a_1)}\otimes\cdots\otimes\widehat{\theta_{\iota_{n-1}}(a_{n-1})}\otimes Q^{\mathrm{o}}_{\iota_n}(\theta_{\iota_n}(a_n)\zeta_1)\otimes\zeta_2\otimes\cdots\otimes\zeta_p\\
&\quad=v_\iota^*\big((a_1\otimes\xi_{\iota_1})\otimes\cdots\otimes(a_{n-1}\otimes\xi_{\iota_{n-1}})\otimes R^{\mathrm{o}}_{\iota_n}(a_n\otimes\zeta_1)\otimes(1\otimes\zeta_2)\otimes\cdots\otimes(1\otimes\zeta_p)\big)\\
&\quad=P_\ell\theta(a_1\cdots a_n)(\zeta_1\otimes\cdots\otimes\zeta_p)
\end{aligned}$$

If $p+n-\ell=2r+1$ for $r\in\{1,2,\ldots,\min(p,n)-2\}$ then both sides of (14) are zero unless $\iota_n=k_1$, $\iota_{n-1}=k_2,\ldots,\iota_{n-r+1}=k_r$, in which case

$$\begin{aligned}
&P_\ell\theta_{\iota_1}(a_1)\cdots\theta_{\iota_n}(a_n)(\zeta_1\otimes\cdots\otimes\zeta_p)=\\
&\quad=\widehat{\theta_{\iota_1}(a_1)}\otimes\cdots\otimes\widehat{\theta_{\iota_{n-r-1}}(a_{n-r-1})}\otimes\\
&\qquad\otimes Q^{\mathrm{o}}_{\iota_{n-r}}\big(\theta_{\iota_{n-r}}(a_{n-r})\langle\xi,\,\theta_{\iota_{n-r+1}}(a_{n-r+1})\cdots\theta_{\iota_n}(a_n)\zeta_1\otimes\cdots\otimes\zeta_r\rangle\zeta_{r+1}\big)\otimes\\
&\qquad\otimes\zeta_{r+2}\otimes\cdots\otimes\zeta_p\\
&\quad=\widehat{\theta_{\iota_1}(a_1)}\otimes\cdots\otimes\widehat{\theta_{\iota_{n-r-1}}(a_{n-r-1})}\otimes\\
&\qquad\otimes Q^{\mathrm{o}}_{\iota_{n-r}}\big(\theta_{\iota_{n-r}}(a_{n-r})\langle\widehat{\theta_{\iota_n}(a_n^*)}\otimes\cdots\otimes\widehat{\theta_{\iota_{n-r+1}}(a^*_{n-r+1})},\,\zeta_1\otimes\cdots\otimes\zeta_r\rangle\zeta_{r+1}\big)\otimes\\
&\qquad\otimes\zeta_{r+2}\otimes\cdots\otimes\zeta_p\\
&\quad=v^*\Big((a_1\otimes\xi_{\iota_1})\otimes\cdots\otimes(a_{n-r-1}\otimes\xi_{\iota_{n-r-1}})\otimes\\
&\qquad\quad\otimes R^{\mathrm{o}}_{\iota_{n-r}}\Big(\theta_{\iota_{n-r}}(a_{n-r})\,\cdot\\
&\qquad\qquad\cdot\big\langle\sigma_{\iota_n}(a_n^*)\cdots\sigma_{\iota_{n-r+1}}(a^*_{n-r+1})\eta,\,(1\otimes\zeta_1)\otimes\cdots\otimes(1\otimes\zeta_r)\big\rangle(1\otimes\zeta_{r+1})\Big)\otimes\\
&\qquad\quad\otimes(1\otimes\zeta_{r+2})\otimes\cdots\otimes(1\otimes\zeta_p)\Big)
\end{aligned}$$

$$
\begin{aligned}
= v^*\Big(& (a_1 \otimes \xi_{\iota_1}) \otimes \cdots \otimes (a_{n-r-1} \otimes \xi_{\iota_{n-r-1}}) \otimes \\
& \otimes R^{\mathrm{o}}_{\iota_{n-r}} \Big(\theta_{\iota_{n-r}}(a_{n-r}) \cdot \\
& \qquad \cdot \big\langle \eta,\, \sigma_{\iota_{n-r+1}}(a_{n-r+1}) \cdots \sigma_{\iota_n}(a_n)(1 \otimes \zeta_1) \otimes \cdots \otimes (1 \otimes \zeta_r) \big\rangle (1 \otimes \zeta_{r+1}) \Big) \otimes \\
& \otimes (1 \otimes \zeta_{r+2}) \otimes \cdots \otimes (1 \otimes \zeta_p) \Big) \\
= P_\ell \theta & (a_1 \cdots a_n)(\zeta_1 \otimes \cdots \otimes \zeta_p).
\end{aligned}
$$

A similar calculation shows that (14) holds also when $n + p - \ell = 2\min(p, n) - 1$.

If $n + p - \ell = 2r$ is even for $r \in \{1, 2, \ldots, \min(p, n) - 1\}$ then both sides of (14) are zero unless $\iota_n = k_1$, $\iota_{n-1} = k_2, \ldots, \iota_{n-r+1} = k_r$ and $\iota_{n-r} \neq k_{r+1}$, in which case

$$
\begin{aligned}
& P_\ell \theta_{\iota_1}(a_1) \cdots \theta_{\iota_n}(a_n)(\zeta_1 \otimes \cdots \otimes \zeta_p) = \\
& \quad = \widehat{\theta_{\iota_1}(a_1)} \otimes \cdots \otimes \widehat{\theta_{\iota_{n-r}}(a_{n-r})} \otimes \\
& \qquad \otimes \langle \xi,\, \theta_{\iota_{n-r+1}}(a_{n-r+1}) \cdots \theta_{\iota_n}(a_n) \zeta_1 \otimes \cdots \otimes \zeta_r \rangle \zeta_{r+1} \otimes \\
& \qquad \otimes \zeta_{r+2} \otimes \cdots \otimes \zeta_p \\
& \quad = \widehat{\theta_{\iota_1}(a_1)} \otimes \cdots \otimes \widehat{\theta_{\iota_{n-r}}(a_{n-r})} \otimes \\
& \qquad \otimes \langle \widehat{\theta_{\iota_n}(a_n^*)} \otimes \cdots \otimes \widehat{\theta_{\iota_{n-r+1}}(a_{n-r+1}^*)},\, \zeta_1 \otimes \cdots \otimes \zeta_r \rangle \zeta_{r+1} \otimes \\
& \qquad \otimes \zeta_{r+2} \otimes \cdots \otimes \zeta_p \\
& \quad = v^* \Big((a_1 \otimes \xi_{\iota_1}) \otimes \cdots \otimes (a_{n-r} \otimes \xi_{\iota_{n-r}}) \otimes \\
& \qquad\quad \otimes \big\langle \eta,\, \sigma_{\iota_{n-r+1}}(a_{n-r+1}) \cdots \sigma_{\iota_n}(a_n) \big((1 \otimes \zeta_1) \otimes \cdots \otimes (1 \otimes \zeta_r) \big) \big\rangle (1 \otimes \zeta_{r+1}) \otimes \\
& \qquad\quad \otimes (1 \otimes \zeta_{r+2}) \otimes \cdots \otimes (1 \otimes \zeta_p) \Big) \\
& \quad = P_\ell \theta(a_1 \cdots a_n)(\zeta_1 \otimes \cdots \otimes \zeta_p).
\end{aligned}
$$

Similar calculations show that (14) holds also when $p + n - \ell = 2\min(p, n)$. This finishes the proof of (11), and of the theorem.

□

§3. Reduced amalgamated free products of von Neumann algebras.

In this section we will describe results for reduced amalgamated free products of von Neumann algebras that are analogous to those for C*–algebras found in §1 and §2. The (reduced) free product of von Neumann algebras with respect to given normal states was defined by Voiculescu in [**27**] and has been much studied. See also Ching's paper [**8**], where the free product of von Neumann algebras with respect to normal faithful tracial states from a certain class was first defined. We begin this section by describing the construction of reduced amalgamated free products of von Neumann algebras (with respect to normal conditional expectations onto von Neumann subalgebras). This construction is only a small step beyond what is in Voiculescu's paper; it has been studied in [**23**], [**24**] and [**12**] in the case of finite von Neumann algebras and trace–preserving conditional expectations, when there is a canonical way to describe the amalgamated free product acting on a Hilbert space.

Following Rieffel [**26**, 5.1], if A and B are von Neumann algebras, if E is a Hilbert B–module and if $\theta : A \to \mathcal{L}(E)$ is a completely positive map, we say that θ is *normal* if for every $\zeta_1, \zeta_2 \in E$, the map $A \ni a \mapsto \langle \zeta_1, \theta(a)\zeta_2 \rangle \in B$ is normal. This coincides with the usual notion of normality when $B = \mathbf{C}$ (in which case E is a Hilbert space). It is clear that if B is a von Neumann subalgebra of a von Neumann algebra A having a normal conditional expectation $\phi : A \to B$ then the GNS representation of A as bounded adjointable operators on the Hilbert B–module $L^2(A, \phi)$ is normal.

Part (i) of the following straightforward lemma was proved in the case of a $*$–homomorphism by Rieffel as part of [**26**, 5.2].

Lemma 3.1. *Let A and B be von Neumann algebras, let E be a Hilbert B–module and suppose that $\theta : A \to \mathcal{L}(E)$ is completely positive map. Let $\mathcal{H}$ be a Hilbert space and let $\tau : B \to \mathcal{L}(\mathcal{H})$ be a normal $*$–representation. Let $\theta \otimes 1$ denote the completely positive map $A \ni a \mapsto \pi(a) \otimes 1 \in \mathcal{L}(E \otimes_\tau \mathcal{H})$ of A into bounded operators on the Hilbert space $E \otimes_\tau \mathcal{H}$. We have:*

(i) *if θ is normal then $\theta \otimes 1$ is normal;*

(ii) *if $\theta \otimes 1$ is normal and if τ is faithful then θ is normal.*

Proof. If $\zeta_1, \zeta_2 \in E$ and $v_1, v_2 \in \mathcal{H}$ then

$$\big\langle \zeta_1 \otimes v_1, (\theta \otimes 1)(x)(\zeta_2 \otimes v_2) \big\rangle = \big\langle v_1, \tau\big(\langle \zeta_1, \theta(x)\zeta_2 \rangle\big) v_2 \big\rangle. \tag{15}$$

If θ is normal then (15) shows that $\theta \otimes 1$ is continuous from the $\sigma(A, A_*)$ topology on A to the weak–operator topology on $\mathcal{L}(E \otimes_\tau \mathcal{H})$, which implies $\theta \otimes 1$ is normal and proves (i).

If $\theta \otimes 1$ is normal then (15) shows that $x \mapsto \tau(\langle \zeta_1, \theta(x)\zeta_2 \rangle)$ is normal. Assuming also τ is faithful, it follows that θ is normal.

□

Also the following application of Lemma 3.1(i) is in [**26**, 5.2].

Lemma 3.2. *Let B be a unital von Neumann subalgebra of a von Neumann algebra A with a normal conditional expectation $\Phi : A \to B$. Let τ be a normal $*$–representation of B on a Hilbert space $\mathcal{H}$. Then the induced representation, $\tau\upharpoonleft^A$, of τ to A with respect to the conditional expectation Φ is normal.*

Proof. By definition, and in the notation of Lemma 3.1, $\tau\upharpoonleft^A = \pi\otimes 1 : A \to \mathcal{L}(L^2(A,\Phi)\otimes_\tau \mathcal{H})$, where π is the GNS representation of A on $L^2(A,\Phi)$.

□

Lemma 3.3. *Let A, B_1 and B_2 be von Neumann algebras and let E_j be a Hilbert B_j–module ($j = 1, 2$). If $\theta : A \to \mathcal{L}(E_1)$ and $\sigma : B_1 \to \mathcal{L}(E_2)$ are normal completely positive maps, then the completely positive map $\theta \otimes 1 : A \to \mathcal{L}(E_1 \otimes_\sigma E_2)$ is normal.*

Proof. Let $\tau : B_2 \to \mathcal{L}(\mathcal{H})$ be a faithful normal $*$–representation of B_2 on a Hilbert space $\mathcal{H}$. Applying Lemma 3.1 in succession we find that $\sigma \otimes 1 : B_1 \to \mathcal{L}(E_2 \otimes_\tau \mathcal{H})$ is normal, $\theta\otimes 1\otimes 1 : A \to \mathcal{L}(E_1 \otimes_\sigma E_2 \otimes_\tau \mathcal{H})$ is normal, and thus $\theta \otimes 1 : A \to \mathcal{L}(E_1 \otimes_\sigma E_2)$ is normal.

□

Now we can define the reduced amalgamated free product of von Neumann algebras, based on Voiculescu's construction of the reduced amalgamated free product of C*–algebras.

Proposition and Definition 3.4. *Let B be a von Neumann algebra, let I be a set and for every $\iota \in I$ let A_ι be a von Neumann algebra containing a copy of B as a unital von Neumann subalgebra and having a normal conditional expectation $\phi_\iota : A_\iota \to B$ whose GNS representation is faithful. Then there is a unique von Neumann algebra A containing B as a unital von Neumann subalgebra, having a normal conditional expectation $\phi : A \to B$ and embeddings (i.e. normal, injective $*$–homomorphisms) $A_\iota \hookrightarrow A$ that restrict to the identity map on B, and such that*

(i) *$\phi\restriction_{A_\iota} = \phi_\iota$, for every $\iota \in I$,*
(ii) *the family $(A_\iota)_{\iota\in I}$ is free with respect to ϕ,*
(iii) *A is the von Neumann algebra generated by $\bigcup_{\iota\in I} A_\iota$,*
(iv) *the GNS representation of ϕ is faithful (on A).*

The resulting pair (A, ϕ) is called the reduced amalgamated free product of von Neumann algebras*, which we will denote by*

$$(A, \phi) = \mathop{\overline{*}}_{\iota\in I} (A_\iota, \phi_\iota).$$

Proof. The idea is to take the free product Hilbert B–module E, on which the C*–algebra reduced amalgamated free product, $\widehat{A}$, of the A_ι's is constructed, to consider a faithful normal representation π of B on a Hilbert space $\mathcal{H}$ and to let A be the closure in strong–operator topology of $\widehat{A}\otimes 1$ acting on the Hilbert space $E\otimes_\pi\mathcal{H}$. It is only required to show that the von Neumann algebra A so obtained is independent of the choice of π.

As in the construction of the reduced amalgamated free product of C*–algebras [**27**], let $E_\iota = L^2(A_\iota,\phi_\iota)$, $\xi_\iota = \widehat{1_{A_\iota}} \in E_\iota$ and let $(E,\xi) = \underset{\iota\in I}{*}(E_\iota,\xi_\iota)$ be the free product of Hilbert B–modules. Then, in light of the observations made at the beginning of this section, the $*$–homomorphisms $\lambda_\iota : A_\iota \to \mathcal{L}(E)$ are seen to be normal. Let $\widehat{A} = C^*\big(\bigcup_{\iota\in I}\lambda_\iota(A_\iota)\big)$, (which is isomorphic to the C*–algebra reduced amalgamated free product), and let $\phi_{\widehat{A}} : \widehat{A}\to B$ be $\phi_{\widehat{A}}(x) = \langle\xi, x\xi\rangle$, which is the conditional expectation arising in the C*–algebra reduced free product construction. Let π be a normal faithful unital representation of B as operators on a Hilbert space $\mathcal{H}$, consider the tensor product representation $\sigma_\pi : x\mapsto x\otimes 1_{\mathcal{H}}$ of $\mathcal{L}(E)$ as operators on the Hilbert space $E\otimes_\pi\mathcal{H}$ and let A_π be the closure in strong–operator topology of $\sigma_\pi(\widehat{A})$ in $\mathcal{L}(E\otimes_\pi\mathcal{H})$. Let P_π be the orthogonal projection from $E\otimes_\pi\mathcal{H}$ onto its subspace $\xi B\otimes_\pi\mathcal{H} = \{\xi\otimes h \mid h\in\mathcal{H}\}\subseteq E\otimes_\pi\mathcal{H}$. Upon identifying $\xi B\otimes_\pi\mathcal{H}$ with $\mathcal{H}$ under the map $\hat{b}\otimes h\mapsto\pi(b)h$, we have $P_\pi(e\otimes h) = \pi(\langle\xi,e\rangle)h$ and if $b\in B$ then $P_\pi\big(\sigma_\pi(b)\big)P_\pi = \pi(b)$. Consider the normal map $\phi_\pi : \mathcal{L}(E\otimes_\pi\mathcal{H})\to\mathcal{L}(\mathcal{H})$ given by $\phi_\pi(y) = P_\pi y P_\pi$. If $x\in\widehat{A}$ then as is seen by an easy calculation, $\phi_\pi\circ\sigma_\pi(x) = \sigma_\pi(\phi_{\widehat{A}}(x))$, which in turn implies that $\phi_\pi(A_\pi) = \pi(B)$.

We shall now show that if π_1 and π_2 are normal faithful representation of B as bounded operators on Hilbert spaces $\mathcal{H}_1$ and respectively $\mathcal{H}_2$, then there is a normal isomorphism $\rho : A_{\pi_1}\to A_{\pi_2}$ such that $\rho\circ\phi_{\pi_1} = \phi_{\pi_2}$. This will allow us to define $A = A_\pi$ and $\phi = \pi^{-1}\circ\phi_\pi : A\to B$. Using [**11**, Theorem 1.4.3] one finds a Hilbert space $\mathcal{H}'$ such that the representations $\pi_1\otimes 1_{\mathcal{H}'}$ and $\pi_2\otimes 1_{\mathcal{H}'}$ are unitarily equivalent. This unitary can be used in the obvious way to define a unitary

$$U : (E\otimes_{\pi_1}\mathcal{H}_1)\otimes\mathcal{H}' = E\otimes_{(\pi_1\otimes 1_{\mathcal{H}'})}(\mathcal{H}_1\otimes\mathcal{H}')\to E\otimes_{(\pi_2\otimes 1_{\mathcal{H}'})}(\mathcal{H}_2\otimes\mathcal{H}') = (E\otimes_{\pi_2}\mathcal{H}_2)\otimes\mathcal{H}'$$

so that $U\circ(\sigma_{\pi_1}(a)\otimes 1_{\mathcal{H}'}) = (\sigma_{\pi_2}(a)\otimes 1_{\mathcal{H}'})\circ U$ for all $a\in\widehat{A}$ and $U\circ(P_{\pi_1}\otimes 1_{\mathcal{H}'}) = (P_{\pi_2}\otimes 1_{\mathcal{H}'})\circ U$. Hence there is a normal $*$–isomorphism $\rho : A_{\pi_1}\to A_{\pi_2}$ such that $\rho(a)\otimes 1_{\mathcal{H}'} = U\circ(a\otimes 1_{\mathcal{H}'})\circ U^*$, and $\rho\circ\phi_{\pi_1} = \phi_{\pi_2}\circ\rho$. Thus the von Neumann algebra $A = A_\pi$ and the normal conditional expectation $\pi^{-1}\circ\phi_\pi : A\to B$ are independent of π.

Properties (i)–(iv) follow from the corresponding facts for the reduced amalgamated free product of C*–algebras, and uniqueness is clear.

□

Here are versions of results in §1 and §2 for W*–algebras.

Lemma 3.5. *Let B be a unital von Neumann algebra, let I be a set and for every $\iota \in I$ let A_ι be a unital von Neumann algebra containing a copy of B as a unital von Neumann subalgebra and having a normal conditional expectation $\phi_\iota : A_\iota \to B$ whose GNS representation is faithful. Let*

$$(A, \phi) = \mathop{\overline{*}}_{\iota \in I} (A_\iota, \phi_\iota)$$

be the reduced amalgamated free product of von Neumann algebras. Then for every $\iota_0 \in I$, there is a normal conditional expectation $\Phi_{\iota_0} : A \to A_{\iota_0}$ such that $\Phi_{\iota_0}\restriction_{A_\iota} = \phi_\iota$ for every $\iota \in I\backslash\{\iota_0\}$ and $\Phi_{\iota_0}(a_1 a_2 \cdots a_n) = 0$ whenever $n \geq 2$ and $a_j \in A_{\iota_j} \cap \ker \phi$ with $\iota_1 \neq \iota_2, \ldots, \iota_{n-1} \neq \iota_n$.

Proof. Let τ be a normal faithful representation of B on a Hilbert space $\mathcal{H}$. The construction of A can be realized on the Hilbert space $E \otimes_\tau \mathcal{H}$. The projection $Q_{\iota_0} : E \to E_{\iota_0}$ from the proof of Lemma 1.1 gives rise to the projection $Q_{\iota_0} \otimes 1_{\mathcal{H}} : E \otimes_\tau \mathcal{H} \to E_{\iota_0} \otimes_\tau \mathcal{H}$, compression with which provides a normal positive linear map $\Theta_{\iota_0} : A \to \mathcal{L}(E_{\iota_0} \otimes_\tau \mathcal{H})$. Let $\lambda_{\iota_0} : A_{\iota_0} \to \mathcal{L}(E_{\iota_0} \otimes_\tau \mathcal{H})$ be the GNS representation $A_{\iota_0} \hookrightarrow \mathcal{L}(E_{\iota_0})$ followed by the inclusion $\mathcal{L}(E_{\iota_0}) \ni x \mapsto x \otimes 1_{\mathcal{H}} \in \mathcal{L}(E_{\iota_0} \otimes_\tau \mathcal{H})$. Then Θ_{ι_0} maps a weakly dense $*$–subalgebra of A into the image of λ_{ι_0}, hence maps all of A there. Let $\Phi_{\iota_0} = \lambda_{\iota_0}^{-1} \circ \Theta_{\iota_0}$. The desired properties of Φ_{ι_0} are easily verified.

□

Theorem 3.6. *Let $\widetilde{B}$ be a von Neumann algebra. Let I be a set and for every $\iota \in I$ let $\widetilde{A}_\iota$ be a von Neumann algebra containing a copy of $\widetilde{B}$ as a unital von Neumann subalgebra and having a normal conditional expectation $\tilde{\phi}_\iota : \widetilde{A}_\iota \to \widetilde{B}$. Suppose that B is a (not necessarily unital) von Neumann subalgebra of $\widetilde{B}$ and that for every $\iota \in I$ A_ι is a von Neumann subalgebra of $\widetilde{A}_\iota$ such that $B \subseteq A_\iota$ and $\tilde{\phi}_\iota(A_\iota) = B$. Let $\phi_\iota : A_\iota \to B$ be the restriction of $\tilde{\phi}_\iota$ and suppose that each of $\tilde{\phi}_\iota$ and ϕ_ι has faithful GNS representation. Let $\kappa_\iota : A_\iota \to \widetilde{A}_\iota$ denote the inclusion. Consider the reduced amalgamated free products of von Neumann algebras*

$$(\widetilde{A}, \tilde{\phi}) = \mathop{\overline{*}}_{\iota \in I} (\widetilde{A}_\iota, \tilde{\phi}_\iota)$$

$$(A, \phi) = \mathop{\overline{*}}_{\iota \in I} (A_\iota, \phi_\iota)$$

and denote the normal inclusions arising from the free product constructions by

$$\widetilde{\alpha}_\iota : \widetilde{A}_\iota \to \widetilde{A}$$

$$\alpha_\iota : A_\iota \to A.$$

Then there is a normal $*$–homomorphism $\kappa : A \to \widetilde{A}$ such that

$$\forall \iota \in I \qquad \kappa \circ \alpha_\iota = \widetilde{\alpha}_\iota \circ \kappa_\iota. \tag{16}$$

Moreover, κ is necessarily injective and is the unique normal $*$–homomorphism satisfying (16).

Proof. This is very much like the proof of Theorem 1.3, to which we refer in detail. Assume without loss of generality that B is a unital subalgebra of $\widetilde{B}$. We now insist that τ be a normal faithful representation of $\widetilde{B}$, and we must show that the algebra homomorphism $\widetilde{\lambda} \circ \widetilde{\sigma} : \mathfrak{A} \to \mathcal{L}(\widetilde{E} \otimes_\tau \mathcal{W})$ extends to a normal representation of the von Neumann algebra A. But $\widetilde{E} \otimes_\tau \mathcal{W}$ is the direct sum of $E \otimes_{\tau\restriction_B} \mathcal{W}$ and the various $\widetilde{\mathcal{W}}(\iota_1, \ldots, \iota_n)$. The homomorphism $\widetilde{\lambda} \circ \widetilde{\sigma}$ restricted to $E \otimes_{\tau\restriction_B} \mathcal{W}$ extends to the defining representation of A. Let $n \geq 1$ and let $\iota_1, \ldots, \iota_n \in I$ be such that $\iota_j \neq \iota_{j+1}$; we have the normal $*$-representation, μ, of A_{ι_1} on the Hilbert space $K_{\iota_1} \otimes_{\widetilde{B}} E_{\iota_2} \otimes_{\widetilde{B}} \cdots \otimes_{\widetilde{B}} E_{\iota_n}$, obtained from the normal representation of A_{ι_1} in $\mathcal{L}(\widetilde{E}_{\iota_1})$; let $\mu\!\downharpoonleft^A$ be the representation of the von Neumann algebra A on $\widetilde{\mathcal{W}}(\iota_1, \ldots, \iota_n)$ induced from μ with respect to the normal conditional expectation $\Phi_{\iota_1} : A \to A_{\iota_1}$ found in Lemma 3.5; then $\widetilde{\lambda} \circ \widetilde{\sigma}$ restricted to $\widetilde{\mathcal{W}}(\iota_1, \ldots, \iota_n)$ extends to the $*$–homomorphism $\mu\!\downharpoonleft^A$, which by Lemma 3.2 is normal.

□

Theorem 3.7. *Let B be a von Neumann algebra, let I be a set and for every $\iota \in I$ let A_ι and D_ι be von Neumann algebras containing copies of B as unital von Neumann subalgebras and having normal conditional expectations $\phi_\iota : A_\iota \to B$, respectively $\psi_\iota : D_\iota \to B$, whose GNS representations are faithful. Suppose that for each $\iota \in I$ there is a normal unital completely positive map $\theta_\iota : A_\iota \to D_\iota$ that is also a B–B bimodule map and satisfies $\psi_\iota \circ \theta_\iota = \phi_\iota$. Let*

$$\begin{aligned} (A, \phi) &= \mathop{\overline{*}}_{\iota \in I} (A_\iota, \phi_\iota) \\ (D, \phi) &= \mathop{\overline{*}}_{\iota \in I} (D_\iota, \psi_\iota) \end{aligned} \tag{17}$$

by the reduced amalgamated free products of von Neumann algebras and denote by $\alpha_\iota : A_\iota \to A$ and $\delta_\iota : D_\iota \to D$ the embeddings arising from the free product constructions. Then there is a normal unital completely positive map $\theta : A \to D$ satisfying

$$\forall \iota \in I \qquad \theta \circ \alpha_\iota = \delta_\iota \circ \theta_\iota \tag{18}$$

and

$$\theta(a_1 a_2 \cdots a_n) = \theta(a_1)\theta(a_2) \cdots \theta(a_n) \tag{19}$$

whenever $a_j \in \alpha_{\iota_j}(A_{\iota_j} \cap \ker \phi_{\iota_j})$ *and* $\iota_1 \neq \iota_2$, $\iota_2 \neq \iota_3$, $\ldots$, $\iota_{n-1} \neq \iota_n$.

Proof. In this proof, we will not denote the von Neumann algebra free products in (17) by (A, ϕ) and (D, ψ). Rather, we will write $(\overline{A}, \overline{\phi}) = \overline{*}_{\iota \in I} (A_\iota, \phi_\iota)$ and $(\overline{D}, \overline{\psi}) = \overline{*}_{\iota \in I} (D_\iota, \psi_\iota)$ for these reduced amalgamated free products of von Neumann algebras; we will reserve the notation $(A, \phi) = *_{\iota \in I} (A_\iota, \phi_\iota)$ and $(D, \psi) = *_{\iota \in I} (D_\iota, \phi_\iota)$ for the reduced amalgamated free products of C*–algebras. Thus $\overline{A}$ and $\overline{D}$ are the closures in strong–operator topology of A and respectively D in the appropriate representations, as specified by Definition 3.4. We need only show that the unital completely positive map $\theta : A \to B$ found in Theorem 2.2 extends to a normal completely positive map $\overline{\theta} : \overline{A} \to \overline{D}$.

Consider, from the proof of Theorem 2.2, the Hilbert B–modules $(E, \xi) = *_{\iota \in I} (E_\iota, \xi_\iota)$, and $(F, \eta) = *_{\iota \in I} (F_\iota, \eta_\iota)$, the $*$–homomorphism $\sigma : A \to \mathcal{L}(F)$ and the bounded operator $v \in \mathcal{L}(E, F)$; denote by i_A the GNS representation of A on $L^2(A, \phi)$. Recall that σ is a free product of embeddings $\sigma_\iota : A_\iota \to \mathcal{L}(F_\iota)$. From the proof of Theorem 1.3, letting τ be a normal faithful representation of B on a Hilbert space $\mathcal{V}$ we see that the representation $\sigma \otimes 1$ of A on $F \otimes_\tau \mathcal{H}$ given by $a \mapsto \sigma(a) \otimes 1$ splits as a direct sum, $\sigma \otimes 1 = \bigoplus_{\lambda \in \Lambda} (\sigma \otimes 1)\restriction_{\mathcal{W}_\lambda}$, where each summand $(\sigma \otimes 1)\restriction_{\mathcal{W}_\lambda}$ is either a copy of $i_A \otimes 1 : A \to \mathcal{L}(L^2(A, \phi) \otimes_\tau \mathcal{H})$ or is the induced representation $\nu \downharpoonleft^A$ of a representation ν of some A_ι on a Hilbert space, where ν is the restriction to an invariant subspace of the representation $\sigma_\iota \otimes 1 : A_\iota \to \mathcal{L}(F_\iota \otimes_\tau \mathcal{H})$. The representation $i_A \otimes 1$ extends to a normal $*$–representation of $\overline{A}$ by the proof of Proposition 3.4. Using Lemma 3.1 we see that $\sigma_\iota \otimes 1$ is normal; hence ν is normal and by Lemma 3.2 $\nu \downharpoonleft^A$ extends to a normal $*$–representation of $\overline{A}$. Hence $\sigma \otimes 1$ extends to a normal $*$–representation of $\overline{A}$, which we will denote by $\overline{\sigma} : \overline{A} \to \mathcal{L}(F \otimes_\tau \mathcal{H})$.

The isometry $v \in \mathcal{L}(E, F)$ gives rise to an isometry $v \otimes 1 : E \otimes_\tau \mathcal{H} \to F \otimes_\tau \mathcal{H}$. Letting $i_D : D \to \mathcal{L}(E)$ be the defining representation, by the proof of Proposition 3.4 the image of $i_D \otimes 1 : D \to \mathcal{L}(E \otimes_\tau \mathcal{H})$ is the von Neumann algebra $\overline{D}$. Consider the normal unital completely positive map $\overline{\theta} : \overline{A} \to \mathcal{L}(E \otimes_\tau \mathcal{H})$ be given by $\overline{\theta}(x) = (v \otimes 1)^* \overline{\sigma}(x)(v \otimes 1)$. If $a \in A$ then $\overline{\theta}(a) = i_D(\theta(a)) \otimes 1$. So $\overline{\theta}$ extends the map $\theta : A \to D$; hence $\overline{\theta}(\overline{A}) \subseteq \overline{D}$.

$\square$

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